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Squared wavefunctions approach to periodic solutions of vector nonlinear Schrödinger equation

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Abstract
We develop a method of squared wavefunctions for the vector nonlinear Schrödinger equation. The squared wavefunctions of the octet representation of $SU(3)$ group give periodic solutions in terms of Weierstrass' elliptic functions. Specific limits of the obtained solution are the plane wave, the soliton and cnoidal waves, which were previously obtained using the ansatz of stationary motion.

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1. Introduction

The motion of light waves in a polarized environment is important for various optical devices. It is described by the coupled nonlinear Schrödinger equation, which was named the Manakov model or the vector nonlinear Schrödinger (VNLS) equation [1–4]. The integrability of the model was first shown by Manakov who also obtained the bright soliton in a focusing medium by applying the inverse scattering method [1]. The application of this type of system to problems of wave physics and nonlinear optics was derived even before Manakov [5]. Recently, there has been considerable interest in the effects of multiple modes, e.g. multifrequency and/or two different polarizations, to dark solitons as well as to bright solitons [6, 7]. The physical importance of the soliton solutions in the Manakov model was widely studied by many authors, leading to the concept of optical switching and soliton-dragging logic gates, etc [8].

Another interesting solution of the VNLS equation as an integrable theory is the periodic solutions. The study of the periodic solutions of an integrable theory has a long history [9–13]. The techniques used were first developed to solve the Korteweg–de Vries equation, which relies upon the important discovery that the solution of the periodic problem could be directly related to algebraic geometry. Nowadays, there are many works on the VNLS equation. Quasi-periodic solutions in terms of N-phase theta functions for the Manakov model are derived in [14], while a series of special solutions is given in [15–18]. In the framework of a special ansatz [19–21] discussed periodic solutions associated with Lamé and Treibich–Verdier potentials for the Manakov model.

The periodic solutions obtained by the finite-band method have rather complicated form which makes it difficult to apply to real physical situations. And there arises the additional problem of extraction of the real solutions [22, 23]. This problem was named the 'effectivization' problem and was solved by a simple modification of the finite-band integration method. This modified method gives solutions in the simplest but important one-phase case [24–26]. The effectivization method was then applied to obtain periodic solutions of various integrable theories associated with the 2×2 linear system [27–29]. It was also applied to describe string configurations of space curve problems by the present author, including the filament motions of fluid mechanics and the vortex motions of the Lund–Regge model [30, 31].

In this paper, we apply the effectivization method to an integrable equation associated with the 3×3 linear system, namely the vector nonlinear Schrödinger equation. To apply the effectivization method to the Manakov model, proper 'squared wavefunctions' must be constructed first. It was explained in section 2 that the adjoint octet representation of the $SU(3)$ group is well suited for this construction. Other irreducible representations only give trivial solutions of the VNLS equation. This fact is explained in the discussion section. Then the general set-up for the periodic problem of the VNLS equation is given in terms of the squared eigenfunctions in section 2. Explicit construction of the lowest phase periodic solution is followed using a modified version of the finite-band integration method in section 3. The resulting formulae involve Weierstrass' elliptic functions. In section 4, various specific solutions from the reduction of periodic solutions are explained in terms of functional relations of Weierstrass' functions. Especially numerical plots as well as explicit check of solutions are done with the help of the symbolic package, Mathematica. Section 5 gives the discussion.

2. The Manakov model

2.1. Linear equation for wavefunctions of fundamental representation

The VNLS equation is an integrable equation,

$$\bar{\partial}\psi_i + \mathrm{i}\partial^2\psi_i + 2\mathrm{i}\big(|\psi_1|^2 + |\psi_2|^2\big)\psi_i = 0 \qquad i = 1, 2 \tag{1}$$

where $\partial \equiv \partial/\partial z$, $\bar{\partial} \equiv \partial/\partial \bar{z}$. The associated $SU(3)$ linear equation of the VNLS equation (Lax pair) is

$$(\partial + E + \lambda T)\Phi = 0 \qquad (\bar{\partial} + E\tilde{E} - \partial\tilde{E} - \lambda E - \lambda^2 T)\Phi = 0 \tag{2}$$

where $T = \mathrm{diag}(2\mathrm{i}/3, -\mathrm{i}/3, -\mathrm{i}/3)$,

$$E = \begin{pmatrix} 0 & \psi_1 & \psi_2 \\ -\psi_1^* & 0 & 0 \\ -\psi_2^* & 0 & 0 \end{pmatrix} \qquad \tilde{E} = \begin{pmatrix} 0 & \mathrm{i}\psi_1 & \mathrm{i}\psi_2 \\ \mathrm{i}\psi_1^* & 0 & 0 \\ \mathrm{i}\psi_2^* & 0 & 0 \end{pmatrix} \tag{3}$$

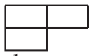
and λ is an arbitrary spectral parameter [1, 32]. It is easy to see that the compatibility of the linear equations,

$$[\partial + E + \lambda T, \bar{\partial} + E\tilde{E} - \partial\tilde{E} - \lambda E - \lambda^2 T] = 0 \tag{4}$$

gives the VNLS equation (1).

The existence of Φ as the solution of the linear equations (2) guarantees that ψ in E is a proper solution of the VNLS equation. In general, the λ dependence of Φ is rather complex, even for the case of 1-soliton. The method of squared wavefunctions relies on the fact that the λ dependence of the *product* of Φ is *polynomial for special cases of quasi-periodic* ψ. The squared wavefunctions should constitute an irreducible representation of $SU(3)$, while Φ constitutes the fundamental triplet representation. Note that irreducible representations of $SU(3)$ can be constructed by proper mixing of symmetric and antisymmetric products of the fundamental representation, which are conventionally represented by Young's tableaux [33].

2.2. *Linear equations for wavefunctions of adjoint representation*

To illustrate the method of squared wavefunctions, we start with the irreducible adjoint representation **8** of $SU(3)$. The multiplet **8**, which is  in terms of Young's tableaux, is obtained by the symmetric product of the fundamental representation such that

$$F_{ijk} = \Phi_i^{(1)}\Phi_j^{(2)}\Phi_k^{(3)} + \Phi_j^{(1)}\Phi_i^{(2)}\Phi_k^{(3)} - \Phi_k^{(1)}\Phi_j^{(2)}\Phi_i^{(3)} - \Phi_k^{(1)}\Phi_i^{(2)}\Phi_j^{(3)}. \tag{5}$$

Here $\Phi_i^{(1)}$, $\Phi_i^{(2)}$, $\Phi_i^{(3)}$ each are solutions of equation (2) which constitute the fundamental **3** representation of $SU(3)$. The eight 'squared' wavefunctions of the **8** multiplet, F_{112}, F_{113}, F_{122}, F_{123}, F_{132}, F_{133}, F_{223}, F_{233}, satisfy the following linear equations,

$$\begin{aligned}
\partial F_{112} &= (E+2A)F_{112} + 2BF_{122} + IF_{113} + 2CF_{132} \\
\partial F_{113} &= (2C-E)F_{133} + HF_{112} + 2BF_{123} + AF_{113} \\
\partial F_{122} &= (A+2E)F_{122} + D/2F_{112} + IF_{123} - C/2F_{223} + IF_{132} \\
\partial F_{123} &= HF_{122} - G/2F_{112} + IF_{133} + BF_{223} + CF_{233} + DF_{113} \\
\partial F_{132} &= HF_{122} + GF_{112} + IF_{12} - B/2F_{223} - 2CF_{233} - D/2F_{113} \\
\partial F_{133} &= -(A+2E)F_{133} + G/2F_{113} + BF_{233} + HF_{123} + HF_{132} \\
\partial F_{223} &= (E-A)F_{223} + 2IF_{233} - 2GF_{122} + 2DF_{123} \\
\partial F_{233} &= DF_{133} - GF_{132} + H/2F_{223} - (E+2A)F_{233}
\end{aligned} \tag{6}$$

where $A = -2\mathrm{i}\lambda/3$, $B = -\psi_1$, $C = -\psi_2$, $D = \psi_1^*$, $E = \mathrm{i}\lambda/3$, $G = \psi_2^*$, $H = I = 0$. The equations for the $\bar\partial$ part are obtained with $\partial \leftrightarrow \bar\partial$ and taking $A = 2\,\mathrm{i}\lambda^2/3 - \mathrm{i}|\psi_1|^2 - \mathrm{i}|\psi_2|^2$, $B = \lambda\psi_1 + \mathrm{i}\partial\psi_1$, $C = \lambda\psi_2 + \partial\psi_2$, $D = -\lambda\psi_1^* + \mathrm{i}\partial\psi_1^*$, $E = -\mathrm{i}\lambda^2/3 + \mathrm{i}|\psi_1|^2$, $G = -\lambda\psi_2^* + \mathrm{i}\partial\psi_2^*$, $H = \mathrm{i}\psi_1\psi_2^*$, $I = \mathrm{i}\psi_1^*\psi_2$. As we will see in the next section, the **8** multiplet gives the quasi-periodic solutions of the VNLS equation.

By explicit substitution of wavefunctions $F_{ijk} = \sum_{n=0}^{m} F_{ijk}^n\lambda^n$ in the linear equations (6), we can obtain the following form of wavefunctions which are consistent with the linear equations of ∂ and $\bar\partial$ part. (This is the simplest one. A more general form will be given in the discussion section.)

$$\begin{aligned}
&F_{112} = 2F_{233}^* = 2\mathrm{i}c_2\psi_2(z,\bar z) \qquad F_{113} = -F_{223}^* = 2\mathrm{i}c_1\psi_1(z,\bar z) \\
&F_{122} = -F_{133}^* = d_0(z,\bar z) \qquad F_{123} = c_1\lambda + d_1(z,\bar z) \qquad F_{132} = c_2\lambda + d_2(z,\bar z)
\end{aligned} \tag{7}$$

where c_1, c_2 are real constants and d_1, d_2 are real functions of $z, \bar z$, while d_0 is a complex function.

Kamchatnov made quasi-periodic solutions more effective by resolving the difficulty due to the fact that the corresponding linear operator L that is not self-adjoint. The 'effectivization' method heavily depends on the invariants of the linear problems. As is well known, the multiplet **8** of $SU(3)$ has two Casimir invariants, C_2 and C_3, which are

$$\begin{aligned} C_2 &= -2F_{112}F_{233} + F_{113}F_{223} + 4F_{133}F_{122} - \tfrac{4}{3}\left(F_{132}^2 + F_{123}^2 + F_{123}F_{132}\right) \\ C_3 &= -\tfrac{3}{2}F_{112}F_{233}(2F_{123} + F_{132}) - \tfrac{3}{4}F_{113}F_{223}(F_{123} + 2F_{132}) - 3F_{133}F_{122}(F_{123} - F_{132}) \\ &\quad + F_{123}^2\left(\tfrac{2}{3}F_{123} + F_{132}\right) - F_{132}^2\left(F_{123} + \tfrac{2}{3}F_{132}\right) + \tfrac{9}{2}F_{113}F_{112}F_{233} + \tfrac{9}{4}F_{112}F_{223}F_{133}. \end{aligned} \tag{8}$$

Using these definitions and the linear equations (6), it can be explicitly checked that C_2 and C_3 are independent of z and $\bar{z}$, and only functions of λ.

3. Periodic solutions

3.1. Q_1, Q_2, Q_3 in terms of s_2, s_4, s_5

As explained in the previous section, the effective method of squared wavefunction begins with two invariants C_2, C_3 of $SU(3)$ [34]. For this, we introduce constants of motion s_i or λ_i, $i = 1$–5, which are defined as

$$\begin{aligned} C_2 &= -\tfrac{4}{3}\left(c_1^2 + c_2^2 + c_1c_2\right)\lambda^2 + s_1\lambda + s_2 = -\tfrac{4}{3}\left(c_1^2 + c_2^2 + c_1c_2\right)(\lambda - \lambda_1)(\lambda - \lambda_2) \\ C_3 &= (c_1 - c_2)\left(\tfrac{5}{3}c_1c_2 + \tfrac{2}{3}c_1^2 + \tfrac{2}{3}c_2^2\right)\lambda^3 + s_3\lambda^2 + s_4\lambda + s_5 \\ &= (c_1 - c_2)\left(\tfrac{5}{3}c_1c_2 + \tfrac{2}{3}c_1^2 + \tfrac{2}{3}c_2^2\right)(\lambda - \lambda_3)(\lambda - \lambda_4)(\lambda - \lambda_5). \end{aligned} \tag{9}$$

Inserting F_{ijk} in equation (7) into equation (8) and identifying it with equation (9), we can obtain expressions of s_i in terms of c_i, d_i, ψ_i. Explicitly, they are

$$\begin{aligned} s_1 &= -\tfrac{4}{3}(2c_1d_1 + 2c_2d_2 + c_1d_2 + c_2d_1) \\ s_2 &= -\tfrac{4}{3}\left(d_1^2 + d_2^2 + d_1d_2 + 3c_1^2Q_1 + 3c_2^2Q_2 + 3R\right) \\ s_3 &= c_1^2(2d_1 + d_2) - c_2^2(d_1 + 2d_2) + 2c_1c_2(d_1 - d_2) \\ &= c_1^2(2d_1 + d_2) + 2c_1c_2d_1 - (1 \leftrightarrow 2\,\text{term}) \\ s_4 &= c_1\left(2d_1^2 - d_2^2 + 2d_1d_2\right) + 3c_1^2(c_1 + 2c_2)Q_1 + 3c_1R - (1 \leftrightarrow 2\,\text{term}) \\ s_5 &= d_1\left(d_1^2 + \tfrac{5}{3}d_1d_2 + d_2^2\right) + 3c_1^2(d_1 + 2d_2)Q_1 + 3d_1R - (1 \leftrightarrow 2\,\text{term}) + 9c_1c_2Q_3 \end{aligned} \tag{10}$$

where $Q_i \equiv |\psi_i(z, \bar{z})|^2, i = 1, 2$, and $Q_3 \equiv \psi_1\psi_2^*d_0 + \psi_1^*\psi_2d_0^*$ and $R \equiv |d_0|^2$. Now by solving for d_1, d_2, Q_1, Q_2 and Q_3 in equation (10), we obtain

$$\begin{aligned} d_1 &= \left(c_1^2 - 2c_1c_2 - 2c_2^2\right)s_1/12c_1c_2(c_1 + c_2) + (c_1 + 2c_2)s_3/9c_1c_2(c_1 + c_2) \\ d_2 &= \left(-2c_1^2 - 2c_1c_2 + c_2^2\right)s_1/12c_1c_2(c_1 + c_2) - (2c_1 + c_2)s_3/9c_1c_2(c_1 + c_2) \\ Q_1 &= (X_1 + s_4/9)\big/c_1^2(c_1 + c_2) - R/c_1(c_1 + c_2) \\ Q_2 &= (X_2 - s_4/9)\big/c_2^2(c_1 + c_2) - R/c_2(c_1 + c_2) \\ Q_3 &= [X_3 - X_4 + (c_1d_2 - c_2d_1)R + (c_1 + c_2)s_5/9 - (d_1 + d_2)s_4/9]/c_1c_2(c_1 + c_2) \end{aligned} \tag{11}$$

where constants X_1, X_2 are

$$\begin{aligned} X_1 &= -(4c_1 - c_2)d_1d_2/9 - (4c_1 + 2c_2)d_1^2\big/9 - (c_1 - c_2)d_2^2\big/9 - (2c_1 + c_2)s_2/12 \\ X_3 &= (4c_1 - 3c_2)d_1^2d_2/9 + (2c_1 - 4c_2)d_2^2\big/27 + c_1d_2s_2/12. \end{aligned} \tag{12}$$

X_2 (X_4) is obtained by exchanging $c_1 \leftrightarrow c_2, d_1 \leftrightarrow d_2$ in X_1 (X_2). Note that this solution is consistent with equation (6).

3.2. Derivation of R

Using equations (6) and (7), the equation for $R = |d_0|^2$ becomes

$$\begin{aligned} \partial R &= \mathrm{i}(c_1 + c_2)(d_0^* \psi_1^* \psi_2 - d_0 \psi_1 \psi_2^*) \\ \bar{\partial} R &= (d_1/c_1 + d_2/c_2)\partial R. \end{aligned} \tag{13}$$

Thus R is a function of $Z \equiv z + (d_1/c_1 + d_2/c_2)\bar{z}$. The first equation of (13) can be rewritten as

$$(\partial R)^2 + (c_1 + c_2)^2 Q_3^2 = (c_1 + c_2)^2 R Q_1 Q_2. \tag{14}$$

With the help of equation (11), equation (14) gives R in terms of Weierstrass' $\wp(u, g_2, g_3)$ function. As far as Weierstrass elliptic functions are involved, we employ the terminology and notation of [35] without further explanations. Explicitly

$$(\partial R)^2 = (c_1 + c_2)^2 \big(4 R Q_1 Q_2 - Q_3^2\big) \equiv \frac{4}{c_1 c_2}(X^3 - g_2 X/4 - g_3/4) \tag{15}$$

where $X \equiv R + \beta$,

$$\begin{aligned} \beta = \frac{c^2}{2916(r+2)}\big[&(126 + 216r^2 + 297r + 45r^3)\tilde{s}_1^2 + (-24r^3 + 120 - 36r^2 + 108r)\tilde{s}_1\tilde{s}_3 \\ &+ (-108r^3 - 756r^2 - 1512r - 864)\tilde{s}_2 + (-720 - 288r + 180r^2 + 72r^3)\tilde{s}_4 \\ &+ (200 + 60r - 48r^2 - 16r^3)\tilde{s}_3^2\big] \end{aligned} \tag{16}$$

$$\begin{aligned} g_2 = \frac{c^4(r+1)}{8748}\big[&-8(r-2)(2r+5)^2\big(6\tilde{s}_3\tilde{s}_5 + 6\tilde{s}_2\tilde{s}_4 - 2\tilde{s}_2\tilde{s}_3^2 - 9\tilde{s}_1\tilde{s}_5 - 2\tilde{s}_4^2 + \tilde{s}_4\tilde{s}_1\tilde{s}_3\big) \\ &+ 9(r+1)^3\big(-4\tilde{s}_2 + \tilde{s}_1^2\big)^2\big] \end{aligned} \tag{17}$$

$$\begin{aligned} g_3 = \frac{c^6}{4251\,528}\big[&(r-2)^2(2r+5)^4\big(-216\tilde{s}_5^2 - 32\tilde{s}_4^3 - 32\tilde{s}_3^3\tilde{s}_5 + 144\tilde{s}_3\tilde{s}_4\tilde{s}_5 + 8\tilde{s}_3^2\tilde{s}_4^2\big) \\ &+ (r-2)(2r+5)^2(r+1)^3\big(-36\tilde{s}_4\tilde{s}_3\tilde{s}_1^3 - 864\tilde{s}_3\tilde{s}_1\tilde{s}_2^2 + 72\tilde{s}_3^2\tilde{s}_1^2\tilde{s}_2 - 864\tilde{s}_4\tilde{s}_1\tilde{s}_5 \\ &+ 648\tilde{s}_1^2\tilde{s}_4\tilde{s}_2 + 864\tilde{s}_2^3 - 864\tilde{s}_3\tilde{s}_5\tilde{s}_2 - 864\tilde{s}_4\tilde{s}_2^2 + 576\tilde{s}_4^2\tilde{s}_2 + 72\tilde{s}_4^2\tilde{s}_1^2 + 576\tilde{s}_3^2\tilde{s}_2^2 \\ &+ 864\tilde{s}_5^2 + 648\tilde{s}_3\tilde{s}_1^2\tilde{s}_5 - 720\tilde{s}_4\tilde{s}_3\tilde{s}_1\tilde{s}_2 + 1296\tilde{s}_1\tilde{s}_5\tilde{s}_2 - 540\tilde{s}_1^3\tilde{s}_5\big) \\ &+ (r+1)^6\big(-324\tilde{s}_2\tilde{s}_1^4 + 1296\tilde{s}_1^2\tilde{s}_2^2 - 1728\tilde{s}_2^3 + 27\tilde{s}_1^6\big)\big] \end{aligned} \tag{18}$$

where $c^2 \equiv c_1c_2, r \equiv c_1/c_2 + c_2/c_1$ and $s_1 = a_1\tilde{s}_1, s_2 = a_1\tilde{s}_2, s_3 = a_2\tilde{s}_3, s_4 = a_2\tilde{s}_4, s_5 = a_2\tilde{s}_5, a_1 = -\frac{4}{3}c^2(r+1), a_2 = \frac{1}{3}c^3\sqrt{r-2}(2r+5)$. Then

$$R = \wp(W + w_3, g_2, g_3) - \beta. \tag{19}$$

Here w_3 is an integration constant and $W = Z/c$. The integration constant w_3 is determined by the initial condition, which we shall choose as follows; $\wp(w_3) = e_3$ at $W = 0$, where e_3 is the smallest root of the equation $4x^3 - g_2x - g_3 = 0$. (The other two roots are denoted by e_1 and e_2 with $e_1 > e_2 > e_3$. w_3 as well as w_1 are called the half period of the $\wp$ function. They satisfy $\wp(w_1) = e_1, \wp(w_2) = e_2, e_1 + e_2 + e_3 = w_1 + w_2 + w_3 = 0$.) This condition is required for R to be real. With this choice, R takes a value between $e_3 - \beta$ and $e_2 - \beta$.

3.3. Derivation of ψ_1, ψ_2

Now we try to obtain the ψ_1 and ψ_2 of the VNLS equation. Using equations (6) and (7), we can obtain the following two equations:

$$\begin{aligned} c_1\partial[\psi_1\exp(-\mathrm{i}d_1z/c_1)] &= -\mathrm{i}d_0^*\psi_2\exp(-\mathrm{i}d_1z/c_1) \\ c_2\partial[\psi_2\exp(-\mathrm{i}d_2z/c_2)] &= \mathrm{i}d_0\psi_1\exp(-\mathrm{i}d_2z/c_2). \end{aligned} \tag{20}$$

Similarly, we can find equations

$$2\mathrm{i}c_1\bar\partial\psi_1 = 2\mathrm{i}c_1(d_1/c_1+d_2/c_2)\partial\psi_1+\eta_1\psi_1 \qquad 2\mathrm{i}c_2\bar\partial\psi_2 = 2\mathrm{i}c_2(d_1/c_1+d_2/c_2)\partial\psi_2+\eta_2\psi_2 \tag{21}$$

where we introduce a constant η_1,

$$\begin{aligned} \eta_1 = \frac{1}{648c_1^3c_2^4}\big[&144(2c_2-c_1)c_1^2c_2^2s_4 - 9\big(c_1^4+c_1^3c_2-6c_1^2c_2^2+2c_1c_2^3+2c_2^4\big)s_1^2 \\ &- 16\big(c_1^2+2c_2^2\big)s_3^2 - 12\big(2c_1^3+c_1^2c_2-2c_1c_2^2-4c_2^3\big)s_1s_3 - 108\big(c_1^2+2c_2^2\big)c_1^2c_2^2s_2\big] \end{aligned} \tag{22}$$

and η_2 is given by substituting $c_1 \leftrightarrow c_2$ and $s_3 \to -s_3, s_4 \to -s_4$ in η_1. They have the solution of the following form,

$$\begin{aligned} &\psi_1 = \sqrt{Q_1(Z)}\exp(\mathrm{i}\theta_1(Z)-\mathrm{i}\eta_1\bar z/2c_1) \qquad \psi_2 = \sqrt{Q_2(Z)}\exp(\mathrm{i}\theta_2(Z)-\mathrm{i}\eta_2\bar z/2c_2) \\ &d_0 = \sqrt{R}\exp(\mathrm{i}\theta_3). \end{aligned} \tag{23}$$

Inserting equation (23) into equation (20), we can find that $\theta_i, i = 1, 2$, satisfies the following differential equation,

$$\frac{\mathrm{d}\theta_1}{\mathrm{d}Z} = -\frac{Q_3}{2c_1Q_1}+\frac{d_1}{c_1} \qquad \frac{\mathrm{d}\theta_2}{\mathrm{d}Z} = \frac{Q_3}{2c_2Q_2}+\frac{d_2}{c_2}. \tag{24}$$

To integrate equation (24), we first introduce constants M_1, M_2, N_1, N_2 such that

$$Q_3/Q_1 = M_1 + N_1/Q_1 \qquad Q_3/Q_2 = M_2+N_2/Q_2 \tag{25}$$

where

$$M_1 = \big[\big(6c_1^3+9c_1^2c_2-9c_1c_2^2-6c_2^3\big)s_1+8\big(c_1^2+c_1c_2+c_2^2\big)s_3\big]\big/36\big(c_1^2c_2^2+c_1c_2^3\big) \tag{26}$$

and M_2 is given by substituting $c_1 \leftrightarrow c_2$ and $s_1 \to -s_1$ in M_1. Explicit form of N_1, N_2 is not required here. Now using equations (19), (24) and (25) and the following identities of the Weierstrass' elliptic function:

$$\int\frac{\mathrm{d}W}{\wp(W)-\wp(\kappa)} = \frac{1}{\wp'(\kappa)}\left\{\ln\frac{\sigma(\kappa-W)}{\sigma(\kappa+W)}+2\zeta(\kappa)W\right\} \tag{27}$$

we can obtain

$$\begin{aligned} \theta_1 &= -\frac{M_1}{2c_1}Z+\frac{\mathrm{i}}{2}\left[\ln\frac{\sigma(\kappa_1-W-\omega_3)}{\sigma(\kappa_1+W+\omega_3)}+2\zeta(\kappa_1)W+\phi_1\right] \\ \theta_2 &= \frac{M_2}{2c_2}Z+\frac{\mathrm{i}}{2}\left[\ln\frac{\sigma(\kappa_2-W-\omega_3)}{\sigma(\kappa_2+W+\omega_3)}+2\zeta(\kappa_2)W+\phi_2\right] \end{aligned} \tag{28}$$

where two constants κ_1, κ_2 are defined by the following two relations:

$$\wp(\kappa_1) = \beta+(X_1+s_4/9)/c_1 \qquad \wp(\kappa_2) = \beta+(X_2-s_4/9)/c_2 \tag{29}$$

$$\wp'(\kappa_{1,2}) = \left.\frac{\mathrm{d}R}{\mathrm{d}W}\right|_{W=\kappa_{1,2}} = \mp\mathrm{i}c(c_1+c_2)Q_3\Big|_{Q_{1,2}=0} = \mp\mathrm{i}c(c_1+c_2)N_{1,2}. \tag{30}$$

By substituting all the above results into equation (23) and using the identity

$$\wp(W) - \wp(\kappa) = -\frac{\sigma(W+\kappa)\sigma(W-\kappa)}{\sigma^2(W)\sigma^2(\kappa)} \tag{31}$$

we finally obtain

$$\begin{aligned}\psi_1 &= \frac{\mathrm{i}}{\sqrt{c_1(c_1+c_2)}}\frac{\sigma(W+\omega_3+\kappa_1)}{\sigma(W+\omega_3)\sigma(\kappa_1)}\exp\left(-\mathrm{i}\frac{M_1}{2c_1}Z + \mathrm{i}\frac{d_1}{c_1}Z - \mathrm{i}\frac{\eta_1}{2c_1}\bar{z} - \zeta(\kappa_1)W - \phi_1\right)\\ \psi_2 &= \frac{\mathrm{i}}{\sqrt{c_2(c_1+c_2)}}\frac{\sigma(W+\omega_3+\kappa_2)}{\sigma(W+\omega_3)\sigma(\kappa_2)}\exp\left(\mathrm{i}\frac{M_2}{2c_2}Z + \mathrm{i}\frac{d_2}{c_2}Z - \mathrm{i}\frac{\eta_2}{2c_2}\bar{z} - \zeta(\kappa_2)W - \phi_2\right).\end{aligned} \tag{32}$$

This is the main result of the present paper. Here, two constants ϕ_1, ϕ_2 are determined as

$$\phi_1 = \zeta(\omega_3)\kappa_1 \qquad \phi_2 = \zeta(\omega_3)\kappa_2 \tag{33}$$

by requiring $Q_i = |\psi_i|^2, i = 1, 2$. Especially at $W = Z = \bar{z} = 0$, we find

$$\begin{aligned}\psi_1(0) &= \frac{\mathrm{i}}{\sqrt{c_1(c_1+c_2)}}\frac{\sigma(\omega_3+\kappa_1)}{\sigma(\omega_3)\sigma(\kappa_1)}\exp(-\phi_1)\\ &= \frac{\mathrm{i}}{\sqrt{c_1(c_1+c_2)}}\sqrt{\wp(\kappa_1)-e_3}\exp[\zeta(\omega_3)\kappa_1 - \phi_1]\end{aligned} \tag{34}$$

where we have used the identity

$$\wp(u) - e_3 = \frac{\sigma^2(u+\omega_3)}{\sigma^2(u)\sigma^2(\omega_3)}\exp\{-2\zeta(\omega_3)u\}. \tag{35}$$

Now using

$$Q_1(0) = \frac{1}{c_1(c_1+c_2)}(\wp(\kappa_1) - e_3) \tag{36}$$

we can obtain the result in equation (33).

3.4. Proper range of λ_i

There is some restriction on the range of parameters λ_i or s_i, which determines the characteristics of the quasi-periodic solutions. From the definition of (9), it is obvious that both λ_1 and λ_2 are real or constitute a complex pair. Similarly, all three $\lambda_i, i = 3, 5$, must be real or one of them is real while the other two constitute a complex pair.

More restrictions on λ_i are related to the definition of $Q_1 = |\psi_1|^2$, $Q_2 = |\psi_2|^2$, $R = |d_0|^2$, such that these three must be positive. Firstly, the discriminant of the Weierstrass equation, $\Delta \equiv g_2^3 - 27g_3^2$, must be positive, which then guarantees $e_i, i = 1, 3$ have real values. In this case, $\wp(W + \omega_3, g_2, g_3)$ takes values between e_3 and e_2. Then using the solution in equation (19), we can see that the condition $R \geqslant 0$ requires $e_3 \geqslant \beta$. Using a similar argument, we can see that the conditions $Q_1 \geqslant 0$, $Q_2 \geqslant 0$ each requires $\wp(\kappa_1) \geqslant e_2$ and $\wp(\kappa_2) \geqslant e_2$. In the following, we call these restrictions the three positivity condition.

The explicit form of Δ is quite complex and here we discuss the characteristics for some specific values of λ_i. Figure 1(*a*) shows the discriminant Δ with λ_3 for $\lambda_4 = -\lambda_5 = 1, \tilde{s}_1 = 2, \tilde{s}_2 = 5, c = 1, r = 3$. This case corresponds to $\lambda_1 = \lambda_2^* = -1 + 2\mathrm{i}$. It shows Δ has a positive value for three regions of (1) $-15.34 \geqslant \lambda_3 \geqslant -9.99$, (2) $-6.83 \geqslant \lambda_3 \geqslant 0.42$ and (3) $14.87 \geqslant \lambda_3 \geqslant 18.94$. But explicit numerical calculation shows that $\wp(\kappa_1)(= -19.90) \leqslant e_2(= -11.03)$ and $\wp(\kappa_2)(= -43.86) \leqslant e_2$ in the region (1) ($\lambda_3 = -15$). Similarly, $\wp(\kappa_2)(= -153.48) \leqslant e_2(= -11.88)$ and $e_3(= -12.49) \geqslant \beta(= -10.56)$ in the region (3) ($\lambda_3 = 15$). Various numerical computations with different values of λ_3 show that these two regions do not satisfy the three positivity conditions. But in the region (2), all the required criteria are satisfied. We find that these features are maintained for various values

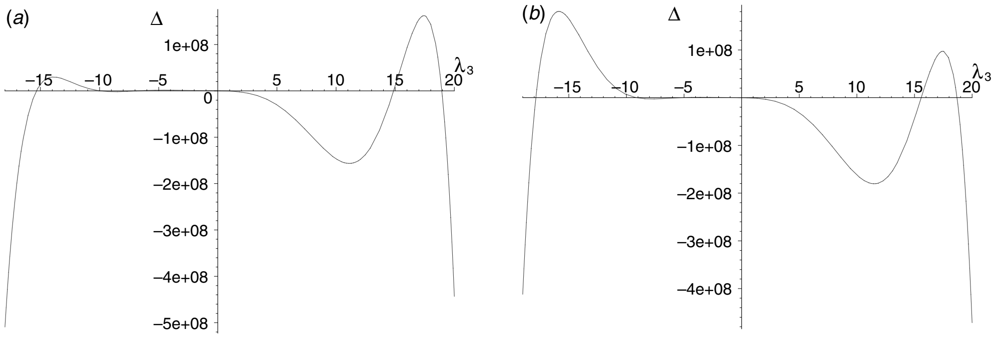


Figure 1. The discriminant Δ with the value λ_3. The parameters for (*a*) are $\lambda_4 = -\lambda_5 = 1, \lambda_1 = \lambda_2^* = -1 + 2\mathrm{i}$. The parameters for (*b*) are $\lambda_4 = -\lambda_5 = \mathrm{i}, \lambda_1 = \lambda_2^* = -1 + 2\mathrm{i}$.

of $\tilde{s}_1, \tilde{s}_2, c, r$. In fact, it can be shown that this feature is maintained when three $\lambda_i, i = 3$–5 are real, which follows due to the following two facts; (i) the discriminant is invariant under the simultaneous shift of $\lambda_i \to \lambda_i +$ constant, (ii) the scaling property of $\Delta \to \rho^{12}\Delta$ under $\lambda_i \to \rho\lambda_i, g_2 \to \rho^4 g_2, g_3 \to \rho^6 g_3$. This property allows us to choose $\lambda_4 = -\lambda_5 = 1$ without changing the essential feature of figure 1.

Figure 1(*b*) shows the discriminant Δ with λ_3 for $\lambda_4 = -\lambda_5 = \mathrm{i}, \tilde{s}_1 = 2, \tilde{s}_2 = 5, c = 1, r = 3$. It shows Δ has a positive value for three regions: (1) $-17.86 \geqslant \lambda_3 \geqslant -9.14$, (2) $-5.23 \geqslant \lambda_3 \geqslant -0.017$ and (3) $15.58 \geqslant \lambda_3 \geqslant 18.69$. Numerical experiment on this case shows only the region (2) of the λ_3 parameter satisfies the three positivity conditions. This case corresponds to λ_3 being real and λ_4, λ_5 constitute a complex pair.

For the case of $\lambda_4 = -\lambda_5 = 1, \tilde{s}_1 = 5, \tilde{s}_2 = 2, r = 3, c = 1$ and $\lambda_4 = -\lambda_5 = \mathrm{i}, \tilde{s}_1 = 5, \tilde{s}_2 = 2, r = 3, c = 1$, numerical computation shows that there does not exist a proper range of λ_3 which satisfies the three positivity conditions. This case corresponds to $\lambda_1, \lambda_2 = (-5 \pm \sqrt{17})/2$. We thus conclude that only a complex pair of λ_1, λ_2 gives the proper quasi-periodic solutions.

4. Special cases

We have seen that a limited range of λ_i values gives the proper quasi-periodic solutions. In this section we study some special cases of the obtained solution by explicitly taking specific λ_i values.

4.1. Plane wave

A simple plane wave solution is obtained by taking $c_2 = -2c_1, \lambda_1 = \lambda_2^* = a_1 + a_2\mathrm{i}$. In this case, any $\lambda_3, \lambda_4, \lambda_5$ value gives the same result. In this case, $g_2 = m^4/3, g_3 = m^6/27, m \equiv c_1 a_2$. And $\Delta = 0, e_1 = m^2/3, e_2 = e_3 = -m^2/6$. The Weierstrass functions are given in a simple form

$$\begin{aligned}
\wp(u) &= -\frac{m^2}{6} + \frac{m^2}{2}\operatorname{cosec}^2(mu/\sqrt{2}) \\
\zeta(u) &= \frac{m^2 u}{6} + \frac{m}{\sqrt{2}}\cot(mu/\sqrt{2}) \\
\sigma(u) &= \frac{\sqrt{2}}{m}\exp(m^2u^2/12)\sin(mu/\sqrt{2}).
\end{aligned} \tag{37}$$

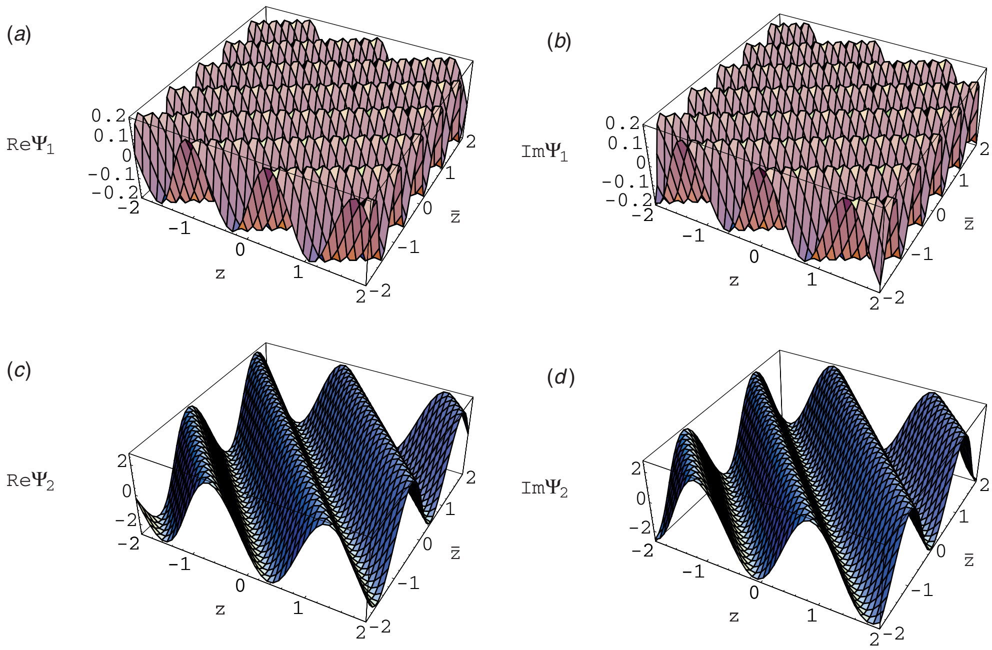


Figure 2. Two-component plane wave. The parameters are $\lambda_3 = -6.83, \lambda_4 = -\lambda_5 = 1, \lambda_1 = \lambda_2^* = -1 + 2\mathrm{i}$.

And $\omega_3 = -\mathrm{i}\infty, \wp(\kappa_1) = -m^2/6, \kappa_1 = -\mathrm{i}\infty, \wp(\kappa_2) = m^2/3, \kappa_2 = \pi/(\sqrt{2}m)$. Now, a straightforward calculation gives $d_1 = -c_1a_1, d_2 = 2c_1a_1, M_1 = M_2 = 0, \eta_1 = 2c_1a_1^2 + 3/2c_1a_2^2, \eta_2 = -4c_1a_1^2 - 2c_1a_2^2, Z = z - 2a_1\bar{z}, W = \mathrm{i}Z/(\sqrt{2}c_1)$. Finally equation (32) gives the quasi-periodic solutions,

$$\psi_1 = 0$$

$$\psi_2 = -\frac{m}{2c_1}\exp\left(-\mathrm{i}a_1z - \mathrm{i}\left(a_1^2 - a_2^2/2\right)\bar{z}\right). \tag{38}$$

In this case, $\wp(W + \omega_3)$ always takes the value $e_2 = e_3$.

More general two-component plane waves can be obtained at the $\Delta = 0$ point of figure 1. As an example, we take $\lambda_3 = -6.83$ in figure 1 with $\lambda_4 = -\lambda_5 = 1, \tilde{s}_1 = 2, \tilde{s}_2 = 5$. Numerical computation gives $g_2 = 363.49, g_3 = 1333.69$, and $e_1 = 11.01, e_2 = e_3 = -5.50, \omega_3 = \mathrm{i}\infty$. And $\kappa_1 = 0.39 + 0.76\mathrm{i}, \kappa_2 = 0.39 + 0.20\mathrm{i}, (-M_1 + 2d_1)/2c_1 = (M_2 + 2d_2)/2c_2 = -0.56, -\eta_1/2c_1 = 2.52, -\eta_2/2c_2 = -6.63$. With $W = Z = z - 1.12\bar{z}$, figures 2(*a*) and (*b*) show the real and imaginary parts of ψ_1. Similarly, figures 2(*c*) and (*d*) plot ψ_2. We use Mathematica to obtain all these plots as well as the necessary numerical values. We also use Mathematica to explicitly check that these $\psi_i, i = 1, 2$, satisfy the VNLS equation (1).

4.2. 1-soliton

An example of the 1-soliton solution is obtained from our quasi-periodic solution when we take $c_1 = c_2 = 1, \lambda_1 = \lambda_2^* = a_1 + a_2\mathrm{i}$. In this case $g_2 = 4a_2^4/3, g_3 = -\frac{8}{27}a_2^6, \Delta = 0$ and

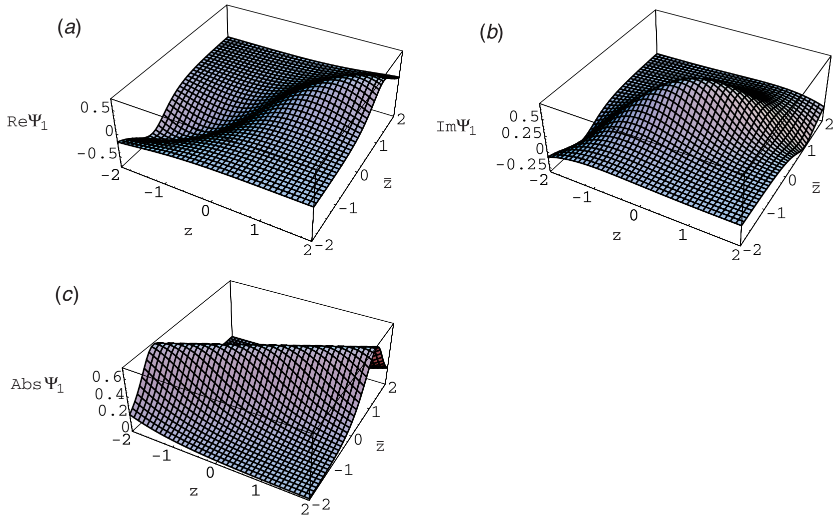


Figure 3. Two-component soliton solution; (*a*) real part, (*b*) imaginary part, (*c*) absolute value of ψ_1 ($=\psi_2$).

$e_1 = e_2 = a_2^2/3, e_3 = -2a_2^2/3, \omega_3 = -\mathrm{i}\pi/(2a_2)$. The Weierstrass functions in this case are given by

$$\begin{aligned} \wp(u) &= -2a_2^2/3 + a_2^2 \coth^2(a_2 u) \\ \zeta(u) &= -a_2^2 u/3 + a_2 \coth(a_2 u) \\ \sigma(u) &= \exp\left(-a_2^2 u^2/6\right) \sinh(a_2 u)/a_2. \end{aligned} \tag{39}$$

And $\wp(\kappa_1) = \wp(\kappa_2) = a_2^2/3, \kappa_1 = \kappa_2 = \infty, M_1 = M_2 = 0, d_1 = d_2 = -a_1, \eta_1 = \eta_2 = 2\left(a_1^2 + a_2^2\right)$. It then gives

$$\psi_1 = \psi_2 = \frac{a_2}{\sqrt{2}} \operatorname{sech}(a_2 Z) \exp\left[-\mathrm{i}a_1 Z - \mathrm{i}\left(a_1^2 + a_2^2\right)\bar{z}\right] \tag{40}$$

where $Z = z - 2a_1\bar{z}$. Figure 3 shows this two-component soliton solution which is obtained by taking $a_1 = a_2 = 1$.

4.3. Solution by Jacobi's elliptic functions

The conventional direct integration method using the stationary ansatz gives some specific quasi-periodic solutions expressed by Jacobi's elliptic functions. These types of solutions are special cases of our solutions. The Weierstrass σ function appearing in the ψ solution in equation (32) can be reduced to Jacobi's elliptic functions under a certain limit. For this, we first note that

$$\frac{\sigma(W + \omega_3 + \kappa_1)}{\sigma(W + \omega_3)} = \exp\left(\frac{\zeta(\omega_1)}{2\omega_1}\left\{\kappa_1^2 + 2\kappa_1(W + w_3)\right\}\right) \frac{H\left(\sqrt{e_1 - e_3}(W + \omega_3 + \kappa_1)\right)}{H\left(\sqrt{e_1 - e_3}(W + w_3)\right)} \tag{41}$$

where $H(u) \equiv \theta_1\left(\frac{\pi}{2K}u\right)$. Note that $\operatorname{cn} u = \sqrt{\frac{k'}{k}}\theta_2\left(\frac{\pi u}{2K}\right)/\theta_4\left(\frac{\pi u}{2K}\right)$, $\operatorname{dn} u = \sqrt{k'}\theta_3\left(\frac{\pi u}{2K}\right)/\theta_4\left(\frac{\pi u}{2K}\right)$ and $\theta_1\left(u + \frac{\pi}{2}\right) = \theta_2(u), \theta_1\left(u + \frac{\pi\theta}{2}\right) = \mathrm{i}q^{-1/4}\,\mathrm{e}^{-\mathrm{i}u}\theta_4(u)$. Here we use the notation of [36]. Thus it is required $\frac{\pi}{2K}\sqrt{e_1 - e_3}\kappa_{1,2} \to \frac{\pi}{2}, \frac{\pi\tau}{2}$ to achieve our goal. In other words, $\kappa_{1,2} \to w_{1,2}$. One such example is given by taking $\lambda_3 = -3, \lambda_4 = -\lambda_5 = 1, \lambda_1 = \lambda_2^* = -1 + 2\mathrm{i}$. In this case,

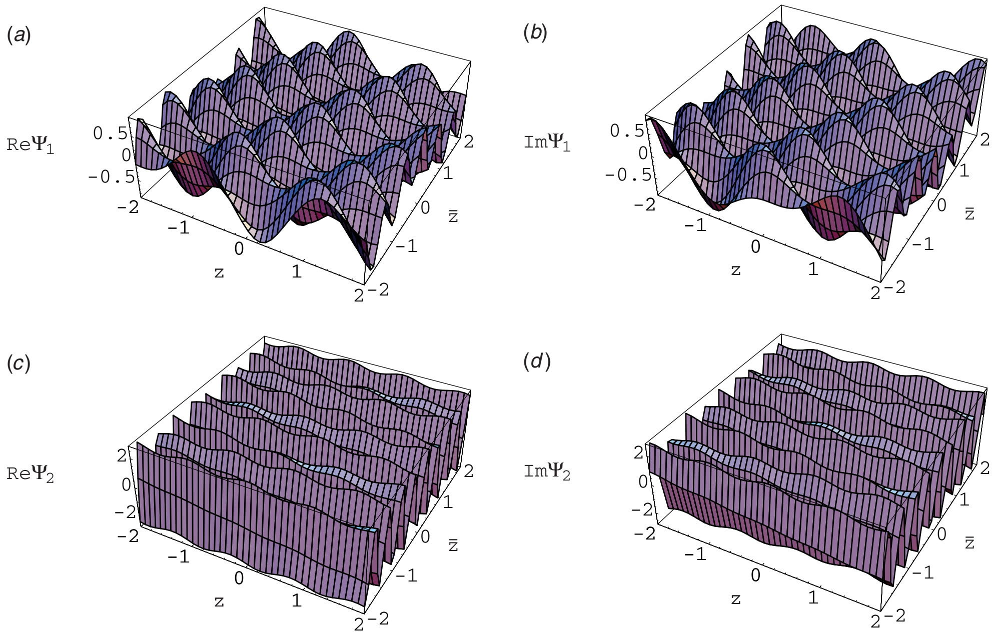


Figure 4. A typical two-component quasi-periodic solution with $\lambda_3 = 4, \lambda_4 = -\lambda_5 = \mathrm{i}, \lambda_1 = \lambda_2^* = -1 + 2\mathrm{i}, c = 1, r = 3$; (*a*) real, (*b*) imaginary part of ψ_1, (*c*) real, (*d*) imaginary part of ψ_2.

$\wp(\kappa_1) = e_2 = -1.46$, $\wp(\kappa_2) = e_1 = 6.16$, $\omega_1 = \kappa_2 = 0.52$, $\omega_3 = 0.63\mathrm{i}$, $\kappa_1 = \omega_1 + \omega_3$. And $g_2 = 271\,360/2187$, $g_3 = 89\,862\,400/531\,441$, $(-M_1 + 2d_1)/2c_1 = (M_2 + 2d_2)/2c_2 = 1$, $-\eta_1/2c_1 = -7.63$, $-\eta_2/2c_2 = -15.25$, $W = Z = z + 2\bar{z}$. Especially, the W-dependent part of ψ_1 becomes

$$\psi_1 \sim \frac{\sigma(W + \omega_3 + \kappa_1)}{\sigma(W + \omega_3)} \exp[-\zeta(\kappa_1)W] \sim \mathrm{cn}(\sqrt{e_1 - e_3}W). \tag{42}$$

In the course of the above derivation, we use the Legendre relation $\zeta(\omega_1)\omega_2 = \zeta(\omega_2)\omega_1 \pm \mathrm{i}\pi/2$. Similarly, we can show $\psi_2 \sim \mathrm{dn}(\sqrt{e_1 - e_3}W)$. All these calculations are numerically checked using Mathematica.

4.4. General quasi-periodic solution

Most generally, the solution in equation (32) describes a two-component quasi-periodic solution. Figure 4 shows one example which is obtained by taking $\lambda_3 = 4, \lambda_4 = -\lambda_5 = \mathrm{i}, \lambda_1 = \lambda_2^* = -1 + 2\mathrm{i}, c = 1, r = 3$. In this case, $g_2 = 244\,256/2187$, $g_3 = 88\,146\,620/531\,441$, $\omega_3 = 0.67\mathrm{i}$, $\kappa_1 = -0.53 - 0.60\mathrm{i}$, $\kappa_2 = 0.53 - 0.05\mathrm{i}$. And $(-M_1 + 2d_1)/2c_1 = (M_2 + 2d_2)/2c_2 = 0.59$, $-\eta_1/2c_1 = -6.26$, $-\eta_2/2c_2 = -13.51$ with $W = Z = z + 1.185\bar{z}$. This is one of the most general configurations described by our periodic solutions.

5. Discussions

In this paper, we apply the method of squared wavefunctions to constructing periodic solutions of the VNLS equation, which improves the effectiveness of the solution. The solution was explicitly given in terms of Weierstrass' elliptic functions. The solution reduces to the already known forms by taking specific values of parameters in the solution. It contains the plane wave, the soliton and the periodic solution expressed in terms of Jacobi's elliptic functions.

In this paper, we use the adjoint **8** representation of $SU(3)$ group to construct the squared wavefunctions. Other irreducible representations of the $SU(3)$ group do not give any interesting solution of the VNLS equation. For example, the multiplet **6**, which is $\boxed{i \mid j}$ in terms of Young's tableaux, is obtained by a symmetric product of the fundamental representation such that

$$F_{ij} = \Phi_i^{(1)}\Phi_j^{(2)} + \Phi_j^{(1)}\Phi_i^{(2)}. \tag{43}$$

There are six independent elements, $F_{11}, F_{12}, F_{13}, F_{22}, F_{23}, F_{33}$, in the multiplet **6**. Then the linear equations (2) on Φ give the following linear equations for F_{ij}:

$$\begin{aligned}
\partial F_{11} &= 2BF_{12} + 2AF_{11} + 2CF_{13} \\
\partial F_{12} &= IF_{13} + DF_{11} + BF_{22} + CF_{23} + (E+A)F_{12} \\
\partial F_{13} &= CF_{33} + GF_{11} - EF_{13} + BF_{23} + HF_{12} \\
\partial F_{22} &= 2IF_{23} + 2DF_{12} + 2EF_{22} \\
\partial F_{23} &= IF_{33} - AF_{23} + GF_{12} + HF_{22} + DF_{13} \\
\partial F_{33} &= 2HF_{23} - 2EF_{33} + 2GF_{13} - 2AF_{33}.
\end{aligned} \tag{44}$$

The notation $A, B, \ldots$ is explained below equation (6). Now, let us assume that F_{ij} are polynomials in λ such that $F_{ij} = \sum_{n=0}^{m} F_{ij}^{n}\lambda^n$ for some integer m. Inserting this expression in equation (44) and considering the coefficient of λ^{m+1}, we can find $F_{ij}^{m} = 0$, which only gives a trivial solution. To avoid this, we need to take an irreducible representation such that part of the linear equations such as in (44) have no λT term. This requires us to take irreducible representations with $3n$ (i.e. 3, 6, 9, ...) boxes in Young's tableaux.

But representations having three boxes in Young's tableaux such as **1** multiplet $\begin{array}{|c|}\hline i \\ \hline j \\ \hline k \\ \hline\end{array}$ and **10** multiplet $\boxed{i \mid j \mid k}$ do not give nontrivial solutions. It is obvious **1** just gives a trivial result. The 10 wavefunctions of the multiplet **10** are constructed as

$$F_{ijk} = \Phi_i^{(1)}\Phi_j^{(2)}\Phi_k^{(3)} + \Phi_j^{(1)}\Phi_i^{(2)}\Phi_k^{(3)} + \Phi_k^{(1)}\Phi_j^{(2)}\Phi_i^{(3)} + (\text{terms with } j \leftrightarrow k \text{ exchanged}). \tag{45}$$

And the linear equations for these wavefunctions can be similarly constructed. Below, we show part of the linear equations which will be required for our argument.

$$\begin{aligned}
\partial F_{111} &= 3CF_{113} + 3AF_{111} + 3BF_{112} \\
\partial F_{112} &= (2A+E)F_{112} + DF_{111} + IF_{113} + 2CF_{123} + 2BF_{122} \\
\partial F_{113} &= HF_{112} + GF_{111} + (A-E)F_{113} + 2CF_{133} + 2BF_{123} \\
\partial F_{122} &= CF_{223} + 2IF_{123} + 2DF_{112} + BF_{222} + (A+2E)F_{122}
\end{aligned} \tag{46}$$

with the same notation as in equation (44). The last equation of (46) shows that this representation has linear equations having no λT term. Thus the multiplet **10** does not have the problem which occurred in the case of **6**. But it only leads to a rather special type of solution, which is essentially that of the single component nonlinear Schrödinger equation. Let F_{ijk} be polynomials in λ such that $F_{ijk} = \sum_{n=0}^{m} F_{ijk}^{n}\lambda^n$ for some integer m. Then equations (46) require that $F_{111}^m = F_{112}^m = F_{113}^m = F_{222}^m = F_{223}^m = F_{233}^m = F_{333}^m = 0$ while $F_{122}^m, F_{123}, F_{133}$ are constants. It then requires $F_{111}^{m-1} = 0$, which results in $F_{112}^{m-1}\psi_1 = -F_{113}\psi_2$, i.e. $F_{122}\psi_1^2 + 2F_{123}\psi_1\psi_2 + F_{133}\psi_2^2 = 0$. Thus $\psi_1 \propto \psi_2$ in this case. Higher multiplets having six or nine boxes in Young's tableaux would also have similar problems.

In section 2, we use equation (7) as the form of F_{ijk} which is consistent with equation (6). In fact, the most general form which is consistent with the equation is

$$\begin{aligned}
F_{112} &= (2\mathrm{i}c_2\psi_2 + 2\mathrm{i}c_0\psi_1)\lambda^{m-1} + \sum_{i=0}^{m-2} F_{112}^i\lambda^i \\
F_{233} &= (-\mathrm{i}c_0^*\psi_1^* - \mathrm{i}c_2\psi_2^*) + \frac{1}{2}\sum_{i=0}^{m-2}\left(F_{112}^i\right)^*\lambda^i \\
F_{113} &= (2\mathrm{i}c_1\psi_1 - 2\mathrm{i}c_0^*\psi_2)\lambda^{m-1} + \sum_{i=0}^{m-2} F_{113}^i\lambda^i \\
F_{223} &= (2\mathrm{i}c_1\psi_1^* - 2\mathrm{i}c_0^*\psi_2^*)\lambda^{m-1} - \sum_{i=0}^{m-2}\left(F_{113}^i\right)^*\lambda^i \\
F_{122} &= c_0\lambda^m + \sum_{i=0}^{m-1} F_{122}^i\lambda^i \qquad F_{133} = -c_0^*\lambda^m - \sum_{i=0}^{m-1}\left(F_{122}^i\right)^*\lambda^i \\
F_{123} &= c_1\lambda^m + \sum_{i=0}^{m-1} F_{123}^i\lambda^i \qquad F_{132} = c_2\lambda^m + \sum_{i=0}^{m-1} F_{132}^i\lambda^i
\end{aligned} \tag{47}$$

where c_0 is a complex constant, c_1, c_2 are real constants, $F_{112}^i, F_{113}^i, F_{122}^i$ are complex functions of $z, \bar{z}$ and F_{123}^i, F_{132}^i are real functions. But this general form does not lead to any new solutions. For example, solutions obtained from the $c_0 = 1$ theory (it means a theory with $c_1 = c_2 = 0, c_0 = 1, m = 1$ in equation (47)) are related to the solutions in equation (32) of $c_0 = 0$ theory in section 3 by the following transformation:

$$\begin{aligned}
&c_0 = 0 \text{ theory} \leftrightarrow c_0 = 1 \text{ theory} \\
&\psi_1 \leftrightarrow \psi_1 + \psi_2 \\
&\psi_2 \leftrightarrow \psi_2 - \psi_1 \\
&\frac{d_1}{c_1} \leftrightarrow \frac{d_2 - d_1 + d_0 + d_0^*}{2} \\
&\frac{d_2}{c_2} \leftrightarrow \frac{-d_2 + d_1 + d_0 + d_0^*}{2} \\
&d_0^* \leftrightarrow -\frac{d_2 + d_1 + d_0^* - d_0}{2}.
\end{aligned} \tag{48}$$

More general theories using the form in equation (47) with $m = 1$ can be shown to be reduced to our solutions of the $c_0 = 0$ theory.

Using the expression in equation (47) with $m \geqslant 2$, we can obtain higher-phase periodic solutions which would be described in terms of Riemann's N-phase theta functions. The difficulty in this program will be the effectivization problem such that the obtained solution should satisfy equation (10). This together with the difficulty in treating Riemann theta functions would be the obstacle to applying it to real physical problems.

The stability analysis of the solution is another important physical problem to be done. A similar study for the case of the one-component nonlinear Schrödinger equation was done in [13]. Especially it gives a method for the analysis of the long-time behaviour of instabilities using the periodic solution of the integrable system.

Acknowledgment

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